Minimal Bell-Kochen-Specker proofs with POVMs on qubits

André Allan Méthot

Département d'informatique et de recherche opérationnelle
Université de Montréal, C.P. 6128, Succ. Centre-Ville
Montréal (QC), H3C 3J7 Canada

methotan@iro.umontreal.ca

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Abstract

There are many different definitions of what a Bell-Kochen-Specker proof with POVMs might be. Here we present and discuss the minimal proof on qubits for three of these definitions and show that they are indeed minimal.

Einstein, Podolsky and Rosen argued in their 1935 paper that quantum mechanics was not complete [1]. Their argument is based on the fact that there seems to be some weird correlations on bipartite measurements of a singlet state, which led Einstein to qualify this phenomenon as "spukhafte Vernwircklungen" (spooky action at a distance). In their opinion, this could only entail the existence of another theory, a local realistic theory, that had variables which where hidden to us. The ignorance of the hidden variables would lead us to think that quantum mechanics is probabilistic and would ensure the Heisenberg uncertainty principle. For almost 30 years, it was debated as to whether such a theory existed and even as to whether this question was a valid question for science—Pauli once compared this question to how many angels can sit on a pin [2]. In 1964, Bell showed that, if the predictions of quantum mechanics are correct and entanglement can resist space-like separation, no realistic local hidden variable theory of quantum

mechanics could exist [3]. Independently, Bell [4] and Kochen and Specker [5] showed that if such a theory existed, it had to be contextual, independently of whether entanglement can resist space-like separation. These proofs will be called BKS theorems hereafter.

In Kochen-Specker paper, they proved that a three level state, or qutrit, could not be described has a non-contextual realistic theory. Such a non-contextual local-hidden-variable (LHV) theory cannot assign a binary value ("yes"/"no") to each element of a measurement such that every von Neumann (projective) measurement on a qutrit would have one and only one element with value "yes" and that this value is independent of the measurement. In other words, once the value "yes"/"no" is assigned, it stays the same for this element in every measurement. Their proof is equivalent to a non-two-coloring proof of a particular graph with 117 vertices where every complete sub-graph of three vertices as one and only one vertex of the first color. However, we know that their proof could not be adapted to a two level state, or qubit, without some modification [4]. In the original BKS proofs, Bell and Kochen-Specker used von Neumann measurements. What if, to show the non-contextuality of a qubit, one requires a more general form a measurements? The most general form is detailed by the positive-operator-valued measure formalism, henceforth called POVM.

We first give the definition of POVMs and contextuality. We then give three definitions of BKS proofs with POVMs, each followed by their minimal proof of the non-contextuality of the qubit and a proof that it is in fact the minimal proof. It is to be noted that there exist many more definitions of what a BKS proof with POVMs might be, see for example [6], and that a consensus as to which is rejects the class of LHV theories that was actually meant by EPR as not been reached.

A POVM is a family of positive matrices $\{M_i\}$ such that $M_i = A_i^\dagger A_i$ and $\sum M_i = \mathbb{1}$, where M_i is called a POVM element. On state ρ, a POVM will output i with probability $\Pr[i] = \mathrm{Tr}(\rho M_i)$ with the state $\frac{1}{\mathrm{Tr}(\rho M_i)} A_i \rho A_i^\dagger$ as the quantum residue. When talking about POVM elements, I will use the notation $\perp$ to denote the inverse of the element: $M + M^\perp = \mathbb{1}$.

Definition 1. *We say that a theory is contextual if the value of a physical operator depends on the context in which it is measured.*

Remark. *Quantum mechanics is* not *a contextual theory.*

When we measure the state ρ with a measurement $\{M_i\}$, we get the result

i with probability $\mathrm{Tr}(\rho M_i)$, which does not depend on the other elements of the measurement M_j for $j \neq i$.

Definition 2. *A BKS proof that quantum mechanics cannot be described by a non-contextual realistic local variable theory can be made if we consider that two measurement elements, which are mathematically identical, are physically equivalent.*

Proposition 1. No local realistic non-contextual description of a qubit exists according to Definition 2.

Proof. In the POVM $\{\mathbb{1}/2, \mathbb{1}/2\}$, one cannot assign one and only one "yes" to an element per POVM since both elements are the same [6, 7]. $\qquad\square$

Theorem 1. *The minimal proof of Proposition 1 requires one POVM of two elements.*

Proof. To prove such a statement, we need to show that such a proof exists and that a smaller one cannot be built. The proof was given above and it is easy in this case to see that a smaller one could not exist. A smaller proof would have one POVM of one element. For such an ensemble, we can always assign one and only one "yes" per POVM since the only POVM we have, $\{\mathbb{1}\}$, has only one element. $\qquad\square$

Definition 3. *A BKS proof that quantum mechanics cannot be described by a non-contextual realistic local variable theory makes sense only if each element in a measurement is considered distinct from the other elements.*

Proposition 2. No local realistic non-contextual description of a qubit exists according to Definition 3.

Proof. The proof of this statement presented here can be found in Cabello's paper [8] and is due to Nakamura. If we consider the following POVMs:

$$
\begin{aligned}
&\{A/2, A^\perp/2, B/2, B^\perp/2\}, \\
&\{A/2, A^\perp/2, C/2, C^\perp/2\} \text{ and} \\
&\{B/2, B^\perp/2, C/2, C^\perp/2\},
\end{aligned}
\tag{1}
$$

since every element appears twice and the number of "yes" needed is odd, 3, one cannot assign non-contextually one and only one "yes" per POVM. $\qquad\square$

Theorem 2. *The minimal proof of Proposition 2 requires three POVMs of four elements each.*

Proof. First let us consider reducing the number of POVMs. If we only have one POVM with distinct elements, it is easy to assign non-contextual "yes"/"no" values to the elements, we choose any element at random to have the "es" value and give the "no" value to all the other elements. For two POVMs, we can select any POVM element that appears twice, or any two POVM elements that appear once, one in the first POVM and the other in the second, to give the "yes" value. It is therefore clear that we need at least three POVMs.

So far we have established that we need three POVMs and that three POVMs with four elements are sufficient. Now let us describe what happens if we reduce the number of elements. If we consider POVMs, with only one element, then it is always the same POVM with the same element, hence it is easy to have a non-contextual description of what happens. POVMs composed of two elements are always composed of an element M and its *unique* inverse $M^\perp$. It is therefore impossible to construct a proof of contextuality.

Let us now examine the case of three POVMs of three elements each. We can express them, without loss of generality, in the form of $\{A_1, B_1, C_1\}$, $\{A_2, B_2, C_2\}$ and $\{A_3, B_3, C_3\}$, such that $A_i \neq B_j$ and $A_i \neq C_j$ for all (i, j). Let us now turn our attention on A_1, A_2 and A_3. Either (1) all these elements are all the same, either (2) two elements of the three elements are the same or either (3) they are all different. In all the three cases we can simply assigne the "yes" value to the A_i elements and the "no" value to the other elements without running into a contradiction.

This as established the we need at least three POVMs, three elements in each POVM and in the special case where we have exactly three POVMs, we require four elements in each POVM. A possibility yet explored would be to construct a proof using four POVMS of three elements each, yielding the same total number of elements. However, such a construction is impossible. Let us write the ensemble in the form $\{A_1, B_1, C_1\}$, $\{A_2, B_2, C_2\}$, $\{A_3, B_3, C_3\}$ and $\{A_4, B_4, C_4\}$. We can always relabel the elements such that either $A_i \neq B_j$ and $A_i \neq C_j$ for all (i, j), or in a way that is isomorphic to the ensemble $\{A_1, B_1, C_1\}$, $\{A_1, B_2, C_2\}$, $\{A_1, B_3, C_3\}$ and $\{B_1, B_2, B_3\}$. In both cases, it is easy to non-contextually assign "yes" values without creating a contradiction. $\qquad\square$

Definition 4. *A BKS proof that quantum mechanics cannot be described by a non-contextual realistic local variable theory must be formulates such that measurements elements M_i of the same POVM are not proportional to one another, $M_j \neq \gamma M_i$ for $j \neq i$ and $\gamma > 0$.*

Proposition 3. No local realistic non-contextual description of a qubit exists according to Definition 4.

Proof. We can use the Cabello's ensemble given in the Equation (1) as we did for Proposition 2. $\square$

Theorem 3. *The minimal proof of Proposition 3 requires three POVMs of four elements each.*

Proof. The proof is exactly the same as the proof of Theorem 2. $\square$

Definition 5. *A BKS proof that quantum mechanics cannot be described by a non-contextual realistic local variable theory has to assign "yes"/"no" values only to the elements of the POVM that could not appear twice in any POVM, $|M_i| > \frac{1}{2}$.*

Toner, Bacon and Ben-Or [7] have a proof that one cannot have a non-contextual description of a qubit according to Definition 5. Their proof has 9 POVMs with 3 to 4 elements each for total of 31 elements.

Although the minimal proofs of Proposition 2 and Proposition 3 are the same, it must be noted that the definitions that led to these proof are fundamentally different. Where Definition 3 only requires the elements to be mathematically distinct for them to be physically different, Definition 4 states that output direction distinguishes physically different measurement elements.

Further work needs to be done to establish what are the minimal proofs for the definition given in [7], but there is a far more fundamental open question. It is not clear which definition of a BKS proof with POVMs is the most natural one. We could argue that Definition 2 is the correct one, that it is simple to show the that quantum mechanics cannot be described by a realistic non-contextual theory and that God indeed plays dice, i.e. Nature is not deterministic. Some might argue that it is evident that using the same elements in a POVM many times will lead to a contradiction and thus Nature assigns indices to elements that appear twice or more in order to make them distinct, in which case something more refined must be used. One can present such arguments for every definition presented here. So the

question remains: What is the definition of a BKS proof with POVMs that is more natural? Maybe such a question is to be left to philosophers, but as Bell, Kochen and Specker showed, one should not make such a claim hastily: there could be some yet unforseen consequence to adopt the *point de vue* of one of the definitions given here.

Since we know that there is a strong link between the conventional BKS proofs and pseudo-telepathy [9, 10], often wrongly called a Bell theorem without inequalities [11], one might wonder why we cannot transform a BKS proof with POVMs on a qubit into a pseudo-telepathy game on an entangled qubit pair [12].

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